

A new exactly solvable quantum model in N dimensions

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Abstract

The N -dimensional position-dependent mass Hamiltonian

$$\hat{H} = \frac{-\hbar^2}{2(1 + \lambda \mathbf{q}^2)} \nabla^2 + \frac{\omega^2 \mathbf{q}^2}{2(1 + \lambda \mathbf{q}^2)}$$

is shown to be exactly solvable for any real positive value of the parameter λ . Algebraically, this Hamiltonian can be thought of as a new maximally superintegrable λ -deformation of the N -dimensional isotropic oscillator and, from a geometric viewpoint, this system is just the intrinsic oscillator potential on an N -dimensional hyperbolic space with nonconstant curvature. The spectrum of this model is shown to be hydrogenlike, and their eigenvalues and eigenfunctions are explicitly obtained by deforming appropriately the symmetry properties of the N -dimensional harmonic oscillator. A further generalization of this construction giving rise to new exactly solvable models is envisaged.

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1 Introduction

The N -dimensional (ND) classical Hamiltonian given by

$$\mathcal{H}(\mathbf{q}, \mathbf{p}) = \mathcal{T}(\mathbf{q}, \mathbf{p}) + \mathcal{U}(\mathbf{q}) = \frac{\mathbf{p}^2}{2(1 + \lambda \mathbf{q}^2)} + \frac{\omega^2 \mathbf{q}^2}{2(1 + \lambda \mathbf{q}^2)}, \quad (1)$$

with real parameters $\lambda > 0$ and $\omega \geq 0$ and where $\mathbf{q}, \mathbf{p} \in \mathbb{R}^N$ are conjugate coordinates and momenta, was proven in [1] to be maximally superintegrable. This means that \mathcal{H} is endowed with the maximum possible number of $(2N - 1)$ functionally independent constants of motion. Explicitly, the integrals of the motion for \mathcal{H} are the ones that encode the radial symmetry of the system, namely,

$$C^{(m)} = \sum_{1 \leq i < j \leq m} (q_i p_j - q_j p_i)^2, \quad C_{(m)} = \sum_{N-m < i < j \leq N} (q_i p_j - q_j p_i)^2, \quad m = 2, \dots, N; \quad (2)$$

together with the functions

$$I_i = p_i^2 - (2\lambda \mathcal{H}(\mathbf{q}, \mathbf{p}) - \omega^2) q_i^2, \quad i = 1, \dots, N. \quad (3)$$

Moreover, each of the three sets $\{\mathcal{H}, C^{(m)}\}$, $\{\mathcal{H}, C_{(m)}\}$ ($m = 2, \dots, N$) and $\{I_i\}$ ($i = 1, \dots, N$) is formed by N functionally independent functions in involution, and the set $\{\mathcal{H}, C^{(m)}, C_{(m)}, I_i\}$ for $m = 2, \dots, N$ with a fixed index i provides the set of $(2N - 1)$ functionally independent functions. Alternatively to the approach performed in [1], both sets of integrals (2) and (3) can also be obtained through a Stäckel transform (or coupling constant metamorphosis) [2, 3, 4, 5, 6] from the free Euclidean motion which has been achieved in [7].

Maximally superintegrable Hamiltonians in N dimensions are quite scarce [8] even on the Euclidean space, and the two representative examples of this class of systems with periodic bounded trajectories are the Kepler system and the isotropic harmonic oscillator. In fact, \mathcal{H} (1) can be interpreted as a genuine (maximally superintegrable) λ -deformation of the ND Euclidean isotropic oscillator with frequency ω , since the limit $\lambda \rightarrow 0$ of (1) yields

$$\mathcal{H}_0 = \frac{1}{2} \mathbf{p}^2 + \frac{1}{2} \omega^2 \mathbf{q}^2.$$

Moreover, an important property of \mathcal{H} is that it can be written as

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^N I_i,$$

and, again, the limit $\lambda \rightarrow 0$ of the integrals I_i transforms this equation into the separability property for the ND harmonic oscillator.

From a geometric perspective, \mathcal{T} can be interpreted as the kinetic energy defining the geodesic motion of a particle with unit mass on a conformally flat space which is the complete Riemannian manifold \mathbb{R}^N , with metric

$$ds^2 = (1 + \lambda \mathbf{q}^2) d\mathbf{q}^2, \quad (4)$$

and nonconstant scalar curvature given by

$$R = -\lambda \frac{(N-1)(N(2+3\lambda\mathbf{q}^2) - 6\lambda\mathbf{q}^2)}{(1+\lambda\mathbf{q}^2)^3}.$$

The scalar curvature $R(r) \equiv R(|\mathbf{q}|)$ is always a *negative* increasing function such that $\lim_{r \rightarrow \infty} R = 0$ and it has a minimum at the origin

$$R(0) = -2\lambda N(N-1),$$

which is exactly the scalar curvature of the ND *hyperbolic space* with negative constant sectional curvature equal to -2λ . In fact, such a curved space is the ND spherically symmetric generalization of the Darboux surface of type III [9, 10], which was constructed in [8, 11]. On the other hand, the central potential \mathcal{U} was proven in [1, 8] to be an “intrinsic” oscillator potential on that Darboux space (we remark that in [1] we considered the Hamiltonian H with $\mathcal{H} = \kappa H/2$ and $\lambda = 1/\kappa$).

For $N = 3$, the potential \mathcal{U} appears in the context of the so called Bertrand space-times [12, 13]. These spaces are certain $(3+1)\text{D}$ static and spherically symmetric Lorentzian spacetimes for which their bounded geodesic motions are all periodic and possess stable circular orbits. Therefore these spaces provide the natural arena for the generalization of the classical Bertrand’s theorem [14] to relativistic spaces of nonconstant curvature, and the maximal superintegrability of the associated 3D Bertrand Hamiltonians has been recently proven in [15]. Again for $N = 3$, \mathcal{H} arises as a particular case of the generalizations of the MIC–Kepler and Taub–NUT systems constructed in [16, 17] (in particular, \mathcal{H} can be recovered by setting $\nu = 1/2$, $a = 1$ and $b = \lambda$).

Alternatively, \mathcal{H} can be interpreted as a position-dependent mass system in which the conformal factor of the metric (4) is identified with the variable mass function

$$m(\mathbf{q}) = 1 + \lambda\mathbf{q}^2.$$

The construction and analysis of the wide range of applications of several position-dependent mass Schrödinger Hamiltonians can be found in, for instance, [18]–[31] (see references therein). In particular, one-dimensional models containing quadratic mass functions are considered in [25, 26] for certain semiconductor heterostructures.

In this letter we solve the quantization problem for \mathcal{H} when $\lambda > 0$. In the next section we introduce the radial effective potential associated to this system. In section 3 we perform the quantization by imposing the existence of the quantum analog of the full set of integrals of the motion. The explicit solution of the spectral problem is presented in section 4. Furthermore, a generalization of this approach in order to obtain new exactly solvable quantum models with position-dependent mass is sketched in the last section.

2 The classical effective potential

The Hamiltonian \mathcal{H} (1) can also be expressed in terms of hyperspherical coordinates r, θ_j , and canonical momenta p_r, p_{θ_j} , ($j = 1, \dots, N-1$). The N hyperspherical coordinates are given by the radial one $r = |\mathbf{q}| \in \mathbb{R}^+$ and $N-1$ angles $\theta_j \in [0, 2\pi)$. In terms of them we have that

$$q_j = r \cos \theta_j \prod_{k=1}^{j-1} \sin \theta_k, \quad 1 \leq j < N, \quad q_N = r \prod_{k=1}^{N-1} \sin \theta_k,$$

where hereafter any product \prod_l^m such that $l > m$ is assumed to be equal to 1. The metric (4) now reads

$$ds^2 = (1 + \lambda r^2)(dr^2 + r^2 d\Omega^2), \quad (5)$$

where $d\Omega^2$ is the metric on the unit $(N-1)$ D sphere

$$d\Omega^2 = \sum_{j=1}^{N-1} d\theta_j^2 \prod_{k=1}^{j-1} \sin^2 \theta_k.$$

Thus the Hamiltonian (1) becomes

$$\mathcal{H}(r, p_r) = \mathcal{T}(r, p_r) + \mathcal{U}(r) = \frac{p_r^2 + r^{-2} \mathbf{L}^2}{2(1 + \lambda r^2)} + \frac{\omega^2 r^2}{2(1 + \lambda r^2)}, \quad (6)$$

where the total angular momentum is given by

$$\mathbf{L}^2 = \sum_{j=1}^{N-1} p_{\theta_j}^2 \prod_{k=1}^{j-1} \frac{1}{\sin^2 \theta_k}.$$

In these coordinates the integrals of motion $C_{(m)}$ (2) adopt a compact form (the remaining $C^{(m)}$ and I_i have more cumbersome expressions):

$$C_{(m)} = \sum_{j=N-m+1}^{N-1} p_{\theta_j}^2 \prod_{k=N-m+1}^{j-1} \frac{1}{\sin^2 \theta_k}, \quad m = 2, \dots, N;$$

so that $C_{(N)} = \mathbf{L}^2$, which is just the second-order Casimir of the $\mathfrak{so}(N)$ -symmetry algebra of any central potential.

The nonlinear radial potential $\mathcal{U}(r)$ is always a *positive* growing function for $\lambda > 0$ (see figure 1) and such that

$$\mathcal{U}(0) = 0, \quad \lim_{r \rightarrow \infty} \mathcal{U}(r) = \frac{\omega^2}{2\lambda}. \quad (7)$$

However, we have to consider the *variable mass contribution* in the whole dynamics and this can be better understood by introducing the radial effective potential. If we apply the canonical transformation defined by

$$P(r, p_r) = \frac{p_r}{\sqrt{1 + \lambda r^2}}, \quad Q(r) = \frac{1}{2} r \sqrt{1 + \lambda r^2} + \frac{\operatorname{arcsinh}(\sqrt{\lambda} r)}{2\sqrt{\lambda}},$$

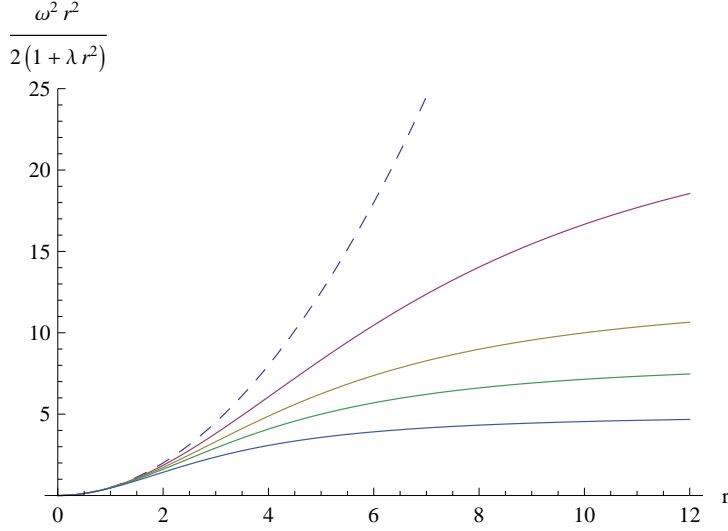


Figure 1: The nonlinear oscillator potential for $\lambda = \{0, 0.02, 0.04, 0.06, 0.1\}$ with $\omega = 1$ is plotted. The upper dashed line corresponds to the isotropic harmonic oscillator with $\lambda = 0$, and the limit $r \rightarrow \infty$ gives $\{+\infty, 25, 12.5, 8.33, 5\}$, respectively.

on the 1D radial Hamiltonian (6), we get

$$\mathcal{H}(Q, P) = \frac{1}{2}P^2 + \mathcal{U}_{\text{eff}}(Q), \quad \mathcal{U}_{\text{eff}}(Q(r)) = \frac{c_N}{2(1 + \lambda r^2)r^2} + \frac{\omega^2 r^2}{2(1 + \lambda r^2)}, \quad (8)$$

where $c_N \geq 0$ is the value of the integral of motion corresponding to the total angular momentum $C_{(N)} \equiv \mathbf{L}^2$. Hence the radial motion for the classical system can be described as a particle moving on a 1D flat space under an effective potential $\mathcal{U}_{\text{eff}}(Q(r))$. The analysis of \mathcal{U}_{eff} shows that, for any value of λ , the energy of the system is always positive and it does have a minimum located at r_{\min} ,

$$r_{\min}^2 = \frac{\lambda c_N + \sqrt{\lambda^2 c_N^2 + \omega^2 c_N}}{\omega^2}, \quad \mathcal{U}_{\text{eff}}(Q(r_{\min})) = -\lambda c_N + \sqrt{\lambda^2 c_N^2 + \omega^2 c_N}, \quad (9)$$

which for the harmonic oscillator ($\lambda = 0$) gives $r_{\min}^2 = \frac{\sqrt{c_N}}{\omega}$, $\mathcal{U}_{\text{eff}}(Q(r_{\min})) = \omega\sqrt{c_N}$.

Now, if $\lambda > 0$ and $c_N \neq 0$, then both $r, Q \in [0, \infty)$ and the effective potential has two representative limits:

$$\lim_{r \rightarrow 0} \mathcal{U}_{\text{eff}}(Q(r)) = +\infty, \quad \lim_{r \rightarrow \infty} \mathcal{U}_{\text{eff}}(Q(r)) = \frac{\omega^2}{2\lambda}, \quad (10)$$

and the latter is just (7). Thus, this effective potential turns out to be hydrogen-like. The values for the minimum of the effective potential, r_{\min} and $\mathcal{U}_{\text{eff}}(Q(r_{\min}))$ (9) are, respectively, greater and smaller than those corresponding to the harmonic oscillator ($\lambda = 0$) case. This effective potential is shown in figure 2.

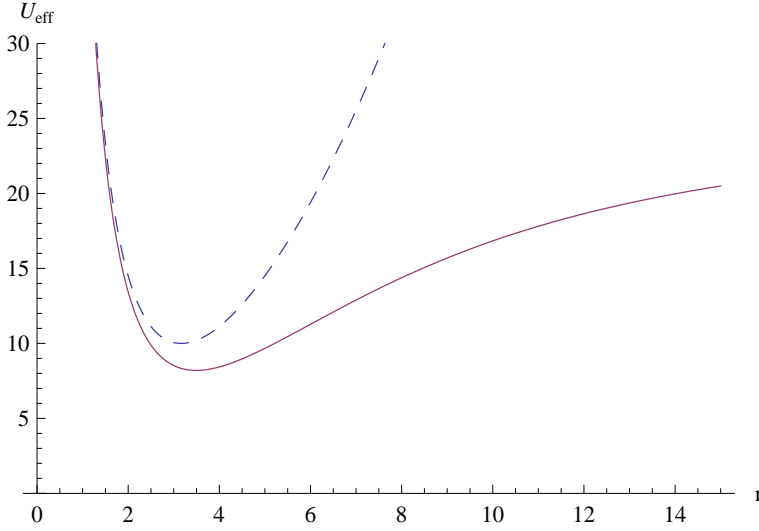


Figure 2: Plot for the effective nonlinear oscillator potential (9) for $\lambda = 0.02$, $c_N = 100$ and $\omega = 1$. The minimum of the potential is located at $r_{\min} = 3.49$ with $\mathcal{U}_{\text{eff}}(r_{\min}) = 8.2$ and $\mathcal{U}_{\text{eff}}(\infty) = 25$. The dashed line corresponds to the effective potential of the harmonic oscillator with $\lambda = 0$ with minimum $\mathcal{U}_{\text{eff}}(r_{\min}) = 10$ at $r_{\min} = 3.16$.

3 A maximally superintegrable quantization

In order to obtain the quantum analogue of the kinetic energy term (1) we have to deal with the unavoidable ordering problems in the canonical quantization process that come from the nonzero curvature of the underlying space (see, e.g. [6]) or, equivalently, from its alternative interpretation as a position-dependent mass Hamiltonian [18]–[31].

A detailed analysis of the different possible quantization prescriptions and a proof of their equivalence through gauge transformations will be presented in [32]. One of these prescriptions consists in the so called “Schrödinger quantization”, which entails that the quantum Hamiltonian $\hat{\mathcal{H}}$ keeps the maximal superintegrability property and is therefore endowed with $2N - 1$ algebraically independent operators that commute with $\hat{\mathcal{H}}$. This prescription allows us to use the full symmetry machinery coming from the harmonic oscillator, and gives rise to the following main result.

Theorem 1. *Let $\hat{\mathcal{H}}$ be the quantum Hamiltonian given by*

$$\hat{\mathcal{H}} = \frac{1}{2(1 + \lambda \hat{\mathbf{q}}^2)} \hat{\mathbf{p}}^2 + \frac{\omega^2 \hat{\mathbf{q}}^2}{2(1 + \lambda \hat{\mathbf{q}}^2)}, \quad (11)$$

such that $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$. For any real value of λ it is verified that

(i) $\hat{\mathcal{H}}$ commutes with the following observables,

$$\hat{C}^{(m)} = \sum_{1 \leq i < j \leq m} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i)^2, \quad \hat{C}_{(m)} = \sum_{N-m < i < j \leq N} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i)^2, \quad m = 2, \dots, N; \quad (12)$$

$$\hat{I}_i = \hat{p}_i^2 - 2\lambda\hat{q}_i^2\hat{\mathcal{H}}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + \omega^2\hat{q}_i^2, \quad i = 1, \dots, N; \quad (13)$$

where $\hat{C}^{(N)} = \hat{C}_{(N)}$ and $\hat{\mathcal{H}} = \frac{1}{2} \sum_{i=1}^N \hat{I}_i$.

(ii) Each of the three sets $\{\hat{\mathcal{H}}, \hat{C}^{(m)}\}$, $\{\hat{\mathcal{H}}, \hat{C}_{(m)}\}$ ($m = 2, \dots, N$) and $\{\hat{I}_i\}$ ($i = 1, \dots, N$) is formed by N algebraically independent commuting observables.

(iii) The set $\{\hat{\mathcal{H}}, \hat{C}^{(m)}, \hat{C}_{(m)}, \hat{I}_i\}$ for $m = 2, \dots, N$ with a fixed index i is formed by $2N - 1$ algebraically independent observables.

(iv) $\hat{\mathcal{H}}$ is formally self-adjoint on the Hilbert space $L^2(\mathbb{R}^N, (1 + \lambda\mathbf{q}^2)d\mathbf{q})$, endowed with the scalar product

$$\langle \Psi | \Phi \rangle_\lambda = \int_{\mathbb{R}^N} \overline{\Psi(\mathbf{q})} \Phi(\mathbf{q}) (1 + \lambda\mathbf{q}^2) d\mathbf{q}.$$

The proof of this result can be obtained through direct computation. Further details can be found in [32]. Next, under the usual differential representation given by

$$\hat{q}_i = q_i, \quad \hat{p}_i = -i\hbar\partial_i = -i\hbar\frac{\partial}{\partial q_i}, \quad \nabla = (\partial_1, \dots, \partial_N),$$

the Hamiltonian (11) leads to the following Schrödinger equation

$$\left(\frac{-\hbar^2}{2(1 + \lambda\mathbf{q}^2)} \nabla^2 + \frac{\omega^2\mathbf{q}^2}{2(1 + \lambda\mathbf{q}^2)} \right) \Psi(\mathbf{q}) = E\Psi(\mathbf{q}), \quad (14)$$

and the change to hyperspherical coordinates yields the equation

$$\frac{1}{2(1 + \lambda r^2)} \left(-\hbar^2 \partial_r^2 - \frac{\hbar^2(N-1)}{r} \partial_r + \frac{\hat{\mathbf{L}}^2}{r^2} + \omega^2 r^2 \right) \Psi(r, \theta) = E\Psi(r, \theta), \quad (15)$$

where $\theta = (\theta_1, \dots, \theta_{N-1})$ and $\hat{\mathbf{L}}^2 \equiv \hat{C}_{(N)}$ is the total quantum angular momentum.

4 Spectrum and eigenfunctions

From the effective potential (8) one should expect that the quantum Hamiltonian (11) would have both a discrete and a continuous spectrum, and this is indeed the case.

By taking into account that $\hat{\mathcal{H}}$ can be defined in terms of the first integrals \hat{I}_i through $\hat{\mathcal{H}} = \frac{1}{2} \sum_{i=1}^N \hat{I}_i$, we find that the discrete spectrum for the Schrödinger equation, $\hat{\mathcal{H}}\Psi(\mathbf{q}) = E\Psi(\mathbf{q})$ (14) can be fully determined by following exactly the same steps as in the *flat* isotropic harmonic oscillator. Let us consider a factorized wave function together with the eigenvalue equations for the operators \hat{I}_i :

$$\Psi(\mathbf{q}) = \prod_{i=1}^N \psi_i(q_i), \quad \frac{1}{2} \hat{I}_i \Psi = \mu_i \Psi, \quad i = 1, \dots, N.$$

Since $\hat{I}_i = \hat{I}_i(\hat{q}_i, \hat{p}_i, \hat{\mathcal{H}})$, we get

$$\frac{1}{2} \hat{I}_i \psi_i(q_i) = \frac{1}{2} (-\hbar^2 \partial_i^2 + (\omega^2 - 2\lambda E) q_i^2) \psi(q_i) = \mu_i \psi(q_i). \quad (16)$$

Therefore if we define the frequency

$$\Omega(E) = \sqrt{\omega^2 - 2\lambda E} \quad \text{whenever} \quad \omega^2 > 2\lambda E, \quad (17)$$

then (16) can be expressed, in a formal manner, as the Schrödinger equation of the one-particle harmonic oscillator

$$\frac{1}{2} (-\hbar^2 \partial_i^2 + \Omega^2 q_i^2) \psi(q_i) = \mu_i \psi(q_i). \quad (18)$$

Since necessarily $\psi(q_i) \in L^2(\mathbb{R}, dq_i)$, the eigenvalue μ_i and the wave function $\psi(q_i)$ turn out to be

$$\begin{aligned} \mu_i &\equiv \mu_i(E, n_i) = \hbar \Omega \left(n_i + \frac{1}{2} \right), \quad n_i = 0, 1, 2, \dots \\ \psi_i(q_i) &\equiv \psi_{n_i}(E, q_i) = A_{n_i} \left(\frac{\beta^2}{\pi} \right)^{1/4} \exp\{-\beta^2 q_i^2 / 2\} H_{n_i}(\beta q_i), \quad \beta = \sqrt{\frac{\Omega}{\hbar}}, \end{aligned} \quad (19)$$

where $H_{n_i}(\beta q_i)$ is the n_i -th Hermite polynomial and A_{n_i} is a normalization constant. Now from

$$\hat{\mathcal{H}}\Psi = \frac{1}{2} \left(\sum_{i=1}^N \hat{I}_i \right) \Psi = \sum_{i=1}^N \mu_i \Psi \equiv E\Psi,$$

we obtain that

$$E = \hbar \Omega \left(n + \frac{N}{2} \right) = \hbar \sqrt{\omega^2 - 2\lambda E} \left(n + \frac{N}{2} \right), \quad n = \sum_{i=1}^N n_i, \quad n = 0, 1, 2, \dots \quad (20)$$

which gives an explicit expression for the energies

$$E \equiv E_n = -\hbar^2 \lambda \left(n + \frac{N}{2} \right)^2 + \hbar \left(n + \frac{N}{2} \right) \sqrt{\hbar^2 \lambda^2 \left(n + \frac{N}{2} \right)^2 + \omega^2}. \quad (21)$$

Note that the degeneracy of this spectrum is exactly the same as in the ND isotropic oscillator, a feature that is again a signature of the maximal superintegrability of the model.

Concerning the continuous spectrum of $\hat{\mathcal{H}}$, it can be rigorously proven [32] that when $\lambda > 0$, it is given by $[\frac{\omega^2}{2\lambda}, \infty)$. Moreover, there are no embedded eigenvalues and the singular spectrum is empty. The explicit form for the wave functions connected with the continuous spectrum can also be given in hyperspherical coordinates: it turns out that their radial factor can be expressed in terms of confluent hypergeometric functions [32].

On the other hand, it can also be proven that $\hat{\mathcal{H}}$ has an infinite number of eigenvalues, all of which are contained in $(0, \frac{\omega^2}{2\lambda})$ and their only accumulation point is $\frac{\omega^2}{2\lambda}$, that is, the bottom of the continuous spectrum.

4.1 Solution in hyperspherical coordinates

Let us now consider the Schrödinger equation (15) expressed in hyperspherical variables. This can be solved by factorizing the wave function in the radial and angular components and by considering the separability provided by the first integrals $\hat{C}_{(m)}$ with eigenvalue equations given by

$$\Psi(r, \theta) = \phi(r)Y(\theta), \quad \hat{C}_{(m)}\Psi = c_m\Psi, \quad m = 2, \dots, N.$$

From it, we obtain that $Y(\theta)$ solves completely the angular part and is written, as expected, as the hyperspherical harmonics

$$\hat{\mathbf{L}}^2 Y(\theta) = \hbar^2 l(l + N - 2)Y(\theta), \quad l = 0, 1, 2, \dots$$

so that l is the quantum number of the angular momentum. The eigenvalues of the operators $\hat{C}_{(m)}$ are related with the $N - 1$ quantum numbers of the angular observables as

$$c_k \leftrightarrow l_{k-1}, \quad k = 2, \dots, N - 1, \quad c_N \leftrightarrow l,$$

that is,

$$Y(\theta) \equiv Y_{c_{N-1}, \dots, c_2}^{c_N}(\theta_1, \theta_2, \dots, \theta_{N-1}) \equiv Y_{l_{N-2}, \dots, l_1}^l(\theta_1, \theta_2, \dots, \theta_{N-1}).$$

Hence the radial Schrödinger equation reduces to

$$\frac{1}{2} \left(-\hbar^2 \left(\frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} - \frac{l(l+N-2)}{r^2} \right) + \Omega^2 r^2 \right) \phi(r) = E\phi(r), \quad (22)$$

where we have introduced the frequency (17). Since this is formally the same radial equation as for the ND isotropic oscillator, we directly obtain the energy spectrum and the radial wave function:

$$E \equiv E_{k,l} = \hbar\Omega \left(2k + l + \frac{N}{2} \right), \quad k = 0, 1, 2, \dots$$

$$\phi(r) \equiv \phi_{k,l}(E, r) = B_{k,l} r^l e^{-\frac{\beta^2 r^2}{2}} L_k^{(l+\frac{N-2}{2})}(\beta^2 r^2), \quad \beta = \sqrt{\frac{\Omega}{\hbar}}, \quad (23)$$

where $L_k^{(l+\frac{N-2}{2})}$ are the associated generalized Laguerre polynomials and $B_{k,l}$ is a normalization constant. Finally, if we introduce the principal quantum number $n = 2k + l$, that is, $E_{k,l} \equiv E_n$ (20) we recover the energy spectrum (21).

5 Generalization

A quite simple but very important remark is worth to be commented. Let us consider any exactly solvable constant-mass Schrödinger problem with Hamiltonian

$$\hat{H} = \frac{-\hbar^2}{2} \nabla^2 + \frac{\omega^2}{2} U(\mathbf{q})$$

such that all the eigenvalues E_n and eigenfunctions ψ_n of the spectral problem $\hat{H} \psi_n = E_n \psi_n$ can be obtained analytically.

Then, it turns out that the exact solvability of the position-dependent mass Hamiltonian \hat{H}_λ defined by

$$\hat{H}_\lambda = \frac{-\hbar^2}{2(1 + \lambda U(\mathbf{q}))} \nabla^2 + \frac{\omega^2 U(\mathbf{q})}{2(1 + \lambda U(\mathbf{q}))},$$

is deeply related to that of \hat{H} , since the Schrödinger problem for \hat{H}_λ reads

$$\frac{-\hbar^2}{2(1 + \lambda U(\mathbf{q}))} \nabla^2 \psi_n^\lambda + \frac{\omega^2 U(\mathbf{q})}{2(1 + \lambda U(\mathbf{q}))} \psi_n^\lambda = E_n^\lambda \psi_n^\lambda,$$

and this equation can be written as

$$\frac{-\hbar^2}{2} \nabla^2 \psi_n^\lambda + \frac{\Omega_\lambda^2}{2} U(\mathbf{q}) \psi_n^\lambda = E_n^\lambda \psi_n^\lambda,$$

which is just the spectral problem for \hat{H} with the new energy-dependent frequency Ω_λ given by

$$\Omega_\lambda = \sqrt{\omega^2 - 2\lambda E_n^\lambda}.$$

Obviously, this spectral problem will present different features depending on the values of Ω_λ , but in any case the exact solvability of \hat{H} provides relevant information in order to get the eigenvalues and eigenfunctions for \hat{H}_λ . We stress that the addition of a nonzero constant to the potential $U(\mathbf{q})$ (here scaled to 1) is essential in this procedure.

Therefore, the position-dependent mass system presented in this paper is the result of taking this λ -deformation approach when $U(\mathbf{q})$ is just the harmonic oscillator potential, thus explaining the maximal superintegrability of the system. Obviously, new ND radial models based on other well-known 1D exactly solvable Hamiltonians can be constructed and will be presented elsewhere.

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